JOURNAL OF APPROXIMATION THEORY **83**, 205–219 (1995)

On the Smoothness of the Metric Projection and Its Application to Proximinality in $L^p(S, X)$

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Received July 1, 1992; accepted in revised form January 5, 1995

We obtain some smoothness properties for the metric projection that hold for a large class of proximinal subspaces. As a consequence of this and various other results, we establish the proximinality of $L^p(S, G)$ in $L^p(S, X)$, $0 < p \leqslant \infty$, for a wide class of proximinal subspaces G in X. Thus generalizing many previously known results. © 1995 Academic Press, Inc.

1. INTRODUCTION

Throughout this paper, (S, Σ, μ) is a finite measure space, X is a Banach space in which the norm is denoted by $\|\cdot\|$, and G is a closed subspace of X. For $0 < p \leqslant \infty$, $L^p(S, X)$ is the space of p-Bochner-integrable functions defined on S with values in X. For $f \in L^p(S, X)$ one has

$$\|f\|_p = \begin{cases} \left[\int_S \|f(s)\|^p \, d\mu\right]^{1/p}; & 0 < p < \infty \\ \text{ess.sup } \|f(s)\|; & p = \infty. \end{cases}$$

A subset F of a Banach space E is said to be proximinal in a subset A of E if, for each $x \in A$, there exists at least one element $y \in F$ such that

$$\|x - y\| = d(x, F) = \inf\{\|x - g\|, g \in F\}.$$

The element y is called a best approximation of x in F.

If G is proximinal in X, then the metric projection is the set valued map $\pi: X \to 2^G$ defined by

$$\pi(x) = \{g \in G: \|x - g\| = d(x, G)\}.$$

If $\pi(x)$ is a singleton for each $x \in X$, we say that G is Chebyshev in X.

Let G be proximinal in X. Several authors investigated sufficient conditions for the proximinality of $L^p(S, G)$ in $L^p(S, X)$, c.f. [1], [2], [3], [4]

0021-9045/95 $12.00

and [6]. Intuition tells one that $L^p(S, G)$ should be proximinal in $L^p(S, X)$ without any additional condition. However, any serious attempt would reveal that the problem has no easy answer.

In this paper, we find some common characteristics (c.f. Theorem 3.1) implied by many existing sufficient conditions for the proximality of $L^p(S, G)$ in $L^p(S, X)$. Furthermore, we show that these common characteristics are sufficient for $L^p(S, G)$ to be proximinal in $L^p(S, X)$, hence leading to a common proof for many previously established results, c.f. Theorems 4.1, 4.2, 4.3 and Remark 4.1. Among these characteristics we find that the mapping $\psi: X \to [0, \infty)$ defined by $\psi(x) = d(0, \pi(x))$ is lower semicontinuous.

Before we continue, we want to mention the fact that we could not find an example of a proximinal subspace G in a Banach space X for which the mapping ψ is not lower semicontinuous.

In section 5 we extend some of the results to "Factor L^p-spaces."

2. Notations and Definitions

This section deals with the notations and definitions that we will adopt.

For $x \in X$ and $A \subset X$ we set

$$d(x, A) = \inf\{\|x - a\|, a \in A\}$$

and

$$\pi(x) = \{g \in G: \|x - g\| = d(x, G)\}, \tag{2.1}$$

where $\pi: X \to 2^G$ is the metric projection.

We also set

$$\hat{G} = \{x \in X: d(x, G) = \|x\|\}, \tag{2.2}$$

$$B = \{g \in G: \|g\| \leqslant 1\}, \tag{2.3}$$

and, for $g \in G$ and $r \in (0, \infty)$,

$$B(g, r) = \{y \in G: \|y - g\| \leqslant r\}. \tag{2.4}$$

Given a function $f: S \to X$ and a subset Y in G, we let $\Phi_f: S \to 2^G$ be defined by

$$\Phi_f(s) = \pi(f(s)), \tag{2.5}$$

ψ: $X \to [0, \infty)$ be defined by

$$\psi(x) = d(0, \pi(x)), \tag{2.6}$$

and ψ_Y: $X \to [0, \infty)$ be defined by

$$\psi_Y(x) = d(Y, \pi(x)) \tag{2.7}$$

where $d(Y, \pi(x)) := \inf\{d(y, g): y \in Y, g \in \pi(x)\}$. We note that

$$\psi = \psi_{\{0\}}. \tag{2.8}$$

We now introduce some definitions to which we refer throughout this paper.

DEFINITION 2.1. We say that G is convexly approximatively compact in X if, for every nonempty closed convex subset A of G and every $x \in X$ with $d(x, A) = d(x, G)$, there exists $g \in A$ such that $\|x - g\| = d(x, G)$.

DEFINITION 2.2 [1]. G is said to have the C-property in X if: Given a closed and separable subspace Y in G, $x \in X$, and $r_1, r_2 \in (0, \infty)$ with $r_1 \geqslant r_2$ and $\|x - g_1\| = d(x, B(0, r_1)) = d(x, Y \cap B(0, r_2))$ for some $g_1 \in B(0, r_1)$, there exists $g_2 \in Y \cap B(0, r_2)$ such that $\|x - g_2\| = \|x - g_1\|$.

DEFINITION 2.3 [6]. Let G be a closed subspace of the Banach space X. The pair (G, X) is said to have property (HV) if there exists a linear topology τ on X such that

(i) X is Hausdorff with respect to τ, and G is τ-closed in X.

(ii) The unit ball in G is τ-compact and metrizable.

(iii) Every τ-compact set in G is proximinal in X.

DEFINITION 2.4. G is said to be locally proximinal in X if the closed unit ball B in G is proximinal in X.

Remark 2.1. G is locally proximinal in X if and only if every closed ball in G is proximinal in X.

3. ON THE SMOOTHNESS OF THE METRIC PROJECTION

First, we establish some properties for the mappings ψ and ψ_Y defined by (2.6) and (2.7) respectively.

Let $\hat{G}$ and B be as in (2.2) and (2.3) respectively. Then we have:

LEMMA 3.1. *Let G be proximinal in X and let Y be a nonempty closed convex subset in G. Then the following statements hold true*:

(i) ψ *is lower semicontinuous if and only if* $\psi_B^{-1}(0)$ *is closed.*

(ii) $Y+\hat{G}\subset\psi_Y^{-1}(0)\subset\overline{Y+\hat{G}}=\overline{\psi_Y^{-1}(0)}$.

Proof. (i) Suppose that ψ is lower semicontinuous and let $x_n\to x$ with $x_n\in\psi_B^{-1}(0)$. Since ψ is lower semicontinuous, we have

$$d(0,\pi(x))\leqslant\liminf d(0,\pi(x_n))\leqslant 1.$$

Therefore $x\in\psi_B^{-1}(0)$ and consequently $\psi_B^{-1}(0)$ is closed. Conversely, suppose that $\psi_B^{-1}(0)$ is closed and let $x_n\to x$, $x\in X$. Also, suppose that $\psi(x)>\liminf\psi(x_n)=p$ and let $a=(1/2)(p+\psi(x))$. Noting that $a>0$ and that $\psi(x/a)=(1/a)\,\psi(x)>1$ we obtain, along a subsequence, that

$$(x_n/a)\to(x/a),\ (x_n/a)\in\psi_B^{-1}(0) \text{ for large } n, \text{ and } (x/a)\notin\psi_B^{-1}(0).$$

Thus contradicting the fact that $\psi_B^{-1}(0)$ is closed.

(ii) The first inclusion follows from the definitions of $\hat{G}$ and ψ_Y.

The equality is a direct consequence of the two inclusions. Now let $x\in\psi_Y^{-1}(0)$. Then, there exists $\{y_n\}\subset Y$ and $\{g_n\}\subset\pi(x)$ such that $\|y_n-g_n\|\to 0$ and, consequently, $y_n+(x-g_n)\to x$. Since $y_n+(x-g_n)\in Y+\hat{G}$, we obtain that $x\in Y+\hat{G}$, hence establishing the claim and completing the proof of the lemma. ∎

COROLLARY 3.1. *Let G be proximinal in X.*
Then the following statements hold true:

(i) *If* $B+\hat{G}$ *is closed then* ψ *is lower semicontinuous.*

(ii) *If* ψ *is lower semicontinuous then*

$$\overline{B+\hat{G}}\subset B(0,1+\varepsilon)+\hat{G}, \qquad \textit{for all} \quad \varepsilon>0.$$

Proof. (i) This follows directly from (ii) and (i) in Lemma 3.1.

(ii) Let $\varepsilon>0$ be given. Since ψ is lower semicontinuous we have, by Lemma 3.1,

$$\overline{B+\hat{G}}=\psi_B^{-1}(0)=\psi^{-1}[0,1]\subset B(0,1+\varepsilon)+\hat{G}. \quad ∎$$

Before stating our main theorem for this section, we establish the following preliminary result:

LEMMA 3.2. *Let G be a proximinal subspace of X and suppose that* $(Y\cap B)+\hat{G}$ *is closed for every closed separable subspace Y of G. Then*

$B+\hat{G}$ is closed and, $Y+\hat{G}$ is closed for every closed separable subspace Y of G.

Proof. Let $\{x_n\}$ be a sequence in $B+\hat{G}$ that converges to some element $x \in X$ and, for each n, let $g_n \in \pi(x_n) \cap B$. Then $Y := \overline{\text{span}\{g_n\}}$ is a closed separable subspace of G and $\{x_n\} \subset (Y \cap B) + \hat{G}$ which is closed. Therefore $x \in (Y \cap B) + \hat{G} \subset B + \hat{G}$. Hence $B+\hat{G}$ is closed.

Now let $\{x_n\}$ be a sequence in $Y+\hat{G}$ that converges to some element $x \in X$ and, for each n, let $g_n \in \pi(x_n) \cap Y$. Then, since $d(x, G)$ is a continuous function of x,

$$\lim \|x - g_n\| = \lim \|x_n - g_n\| = \lim d(x_n, G) = d(x, G).$$

Therefore $\{g_n\}$ is bounded and consequently there exists $r > 0$ such that $\{g_n\} \subset B(0, r) \cap Y$. This, together with the fact that $B(0, r) = rB(0, 1) := rB$, $Y = rY$ and $\hat{G} = r\hat{G}$, implies that

$$x_n \in [Y \cap B(0, r)] + \hat{G} = r[(Y \cap B) + \hat{G}].$$

But $(Y \cap B) + \hat{G}$ (hence $r[(Y \cap B) + \hat{G}]$) is closed. Therefore

$$x \in [Y \cap B(0, r)] + \hat{G} \subset Y + \hat{G}$$

and consequently $Y+\hat{G}$ is closed. This ends the proof of the lemma. ∎

We now proceed with our main result for this section:

THEOREM 3.1. *Let G be a proximinal subspace of X. Then any of the following conditions is sufficient in order that ψ be lower semicontinuous and $B+\hat{G}$ be closed:*

(1) *G is convexly approximatively compact in X.*

(2) *G is reflexive.*

(3) *G has the C-property in X.*

(4) $\overline{\text{span}\,\hat{G}}$ *is reflexive.*

(5) *G is locally proximinal in X.*

(6) *X is a dual space and G is wk*-closed in X.*

(7) *The pair (G, X) has property (HV).*

Moreover, if any of conditions (1)–(4) holds then $Y+\hat{G}$ is closed for every closed separable subspace Y of G.

Proof. We divide the proof into two parts:

Part I. If any of conditions (1)–(4) hold: It is sufficient to prove that $(Y \cap B) + \hat{G}$ is closed for every closed separable subspace Y of G. Indeed, for in that case we obtain by Lemma 3.2. that $B+\hat{G}$ is closed (hence by Corollary 3.1. ψ is lower semicontinuous) and $Y+\hat{G}$ is closed for every closed separable subspace Y of G.

So let Y be a closed separable subspace of G and let $\{x_n\}$ be a sequence in $(Y\cap B)+\hat{G}$ that converges to some element $x\in X$. We need to show that $x\in(Y\cap B)+\hat{G}$ or, equivalently, that $\pi(x)\cap(Y\cap B)\neq\emptyset$. For each n, let $g_n\in\pi(x_n)\cap(Y\cap B)$. Then

$$\|x-g_n\|\to d(x, G) \tag{3.1}$$

and consequently, since $\{g_n\}\subset(Y\cap B)\subset G$,

$$d(x, Y\cap B)=d(x, G). \tag{3.2}$$

Hence, if condition (1) holds then, since $Y\cap B$ is a nonempty closed convex subset of G, $\pi(x)\cap(Y\cap B)\neq\emptyset$. This completes the proof for condition (1).

Suppose that condition (2) holds. Then, by [8, Corollary 2.2, p. 384], every nonempty closed convex subset of G is proximinal. Hence condition (1) holds and consequently the proof for condition (2) is complete.

Now let $g_1\in\pi(x)$ and let $r=\max\{1, 2\,\|x\|\}$. Then, by (3.2) and [8, p. 140], we have

$$g_1\in B(0, r) \qquad \text{and} \qquad d(x, Y\cap B)=d(x, B(0, r))=\|x-g_1\|=d(x, G).$$

Hence, if condition (3) holds then, since $B=B(0, 1)$ and $1\leqslant r$, there exists $g_2\in Y\cap B$ such that $\|x-g_2\|=\|x-g_1\|=d(x, G)$. This implies that $g\in\pi(x)\cap(Y\cap B)$ and consequently $\pi(x)\cap(Y\cap B)\neq\emptyset$. This ends the proof for condition (3).

Finally, suppose that condition (4) holds. Since $\{x_n-g_n\}\subset\hat{G}\subset\overline{\operatorname{span}\hat{G}}$ which is reflexive, there exists a subsequence $\{x_{n_k}-g_{n_k}\}$ that converges weakly to an element of X. But x_{n_k} converges strongly (hence weakly) to x and $\{g_{n_k}\}\subset(Y\cap B)$ which is closed and convex hence weakly closed in X, [10, p. 111]. Therefore $\{g_{n_k}\}$ converges weakly to an element $g\in(Y\cap B)$. This, with (3.1) and the fact that $\|\cdot\|$ is weakly lower semicontinuous, [5, p. 345], implies that $g\in\pi(x)\cap(Y\cap B)$ and consequently $\pi(x)\cap(Y\cap B)\neq\emptyset$. This finishes the proof for condition (4).

The proof of part I is now complete.

Part II. If any of conditions (5)–(7) hold: In this case it is sufficient to prove that, if condition (5) holds then $B+\hat{G}$ is closed. Indeed: It follows immediately from (ii) and (iii) of Definition 2.3 that if condition (7) holds then condition (5) holds. Also by [8, Corollary 2.2, p. 384], it follows that if condition (6) holds then conditions (5) holds. Finally, if $B+\hat{G}$ is closed then, by Corollary 3.1, ψ is lower semicontinuous.

So, suppose that condition (5) holds and let $\{x_n\}$ be a sequence in $B+\hat{G}$ that converges to some element $x \in X$. Then $\psi(x_n) \cap B \neq \varnothing$. Consequently we have, since $d(x, B)$ and $d(x, G)$ are continuous functions of x,

$$d(x, B) = \lim d(x_n, B) = \lim d(x_n, G) = d(x, G).$$

This, together with condition (5), implies that there exists $g \in B$ such that

$$\|x-g\| = d(x, B) = d(x, G).$$

Hence $g \in \pi(x) \cap B$ and consequently $x \in B+\hat{G}$. Therefore $B+\hat{G}$ is closed.

This completes the proof of the theorem. ∎

4. Proximinality of $L^p(S, G)$ in $L^p(S, X)$

In this section we prove that, for $0 < p \leqslant \infty$, $L^p(S, G)$ is proximinal in $L^p(S, X)$ whenever G satisfies certain conditions. In the remainder of this paper (S, Σ, μ) is a finite measure space.

We start with the following definitions:

Definition 4.1 [7]. A function $f: S \to X$ is said to be simple if its range contains only finitely many points $x_1, x_2, ..., x_n$ in X, and if $f^{-1}(x_i)$ is measurable for $i = 1, 2, ..., n$.

Definition 4.2 [7]. A function $f: S \to X$ is said to be strongly measurable if there exists a measurable subset N in S of measure zero ($\mu(N) = 0$), and a sequence $\{f_n\}$ of simple functions such that for every $s \in S \backslash N$ we have

$$\|f_n(s) - f(x)\| \to 0.$$

Definition 4.3 [7]. A set valued mapping $\Phi: S \to 2^E$, where E is a subset of X, is said to be weakly measurable if $\Phi^{-1}(\theta)$ is measurable in S whenever θ is open in E. Where $\Phi^{-1}(\theta) = \{s \in S: \Phi(s) \cap \theta \neq \varnothing\}$. We note that if Φ is single-valued, then the weak measurability is equivalent to the measurability in the classical sense.

We now state our result that will allow us to get rid of the separability assumption on G.

Lemma 4.1. *Suppose that G is proximinal in X and that, for every closed and separable subspace Y of G, $Y+\hat{G}$ is closed. Then, given any strongly measurable function $f: S \to X$, there exists a closed and separable subspace Y_f of G such that*

$$\pi(f(s)) \cap Y_f \neq \varnothing \qquad \textit{for a.e.} \quad s \in S.$$

Proof. Since f is strongly measurable, it follows, by [7, p. 115], that range (f) is essentially separable, i.e. there exists a null set $N(\mu(N)=0)$ in S such that $f(S\backslash N)$ is separable. Let $\{x_n\}$ be a dense sequence in $f(S\backslash N)$ and, for each n, let $g_n \in \pi(x_n)$. Then $Y_f = \overline{\operatorname{span}\{g_n\}}$ is a closed and separable subspace of G an consequently, by assumption, $Y_f + \hat{G}$ is closed. Also, since $g_n \in \pi(x_n) \cap Y_f$ for every n, we have $\{x_n\} \subset Y_f + \hat{G}$. Therefore, since $\{x_n\}$ is dense in $f(S\backslash N)$ and $Y_f + \hat{G}$ is closed, we obtain that

$$f(S\backslash N) \subset \overline{\{x_n\}} \subset Y_f + \hat{G}$$

and, consequently,

$$\pi(f(s)) \cap Y_f \neq \emptyset \qquad \text{for a.e.} \quad s \in S. \quad \blacksquare$$

The following lemma is needed for the proof of Theorem 4.1.:

LEMMA 4.2. *Let G be a proximinal subspace of X. Suppose that there exists a positive constant $R \geqslant 1$ such that*

$$\overline{B + \hat{G}} \subset B(0, R) + \hat{G}.$$

Then, for every $g \in G$ and every $r \in (0, \infty)$, we have

$$\overline{B(g, r) + \hat{G}} \subset B(g, Rr) + \hat{G}.$$

Proof. Let $r > 0$ be given. Since

$$B(g, r) = g + rB(0, 1) = g + rB \qquad \text{and} \qquad \hat{G} = r\hat{G}, \tag{4.0}$$

we have

$$B(g, r) + \hat{G} = g + rB + r\hat{G} = g + r(B + \hat{G}).$$

Hence we obtain, from (4.0) and the assumption, that

$$\overline{B(g, r) + \hat{G}} = g + r\overline{(B + \hat{G})} \subset g + r(B(0, R) + \hat{G} = B(g, Rr) + \hat{G}. \quad \blacksquare$$

We are now ready to state and prove our result concerning the proximinality of $L^p(S, G)$ in $L^p(S, X)$:

THEOREM 4.1. *Let G be a proximinal subspace of the Banach space X and $\pi(x)$ be the set of best approximations to x in G. Then for $0 < p \leqslant \infty$, $L^p(S, G)$ is proximinal $L^p(S, X)$, if the following two conditions hold:*

(1) *There exists a positive constant $R \geqslant 1$ such that*

$$\overline{B + \hat{G}} \subset B(0, R) + \hat{G}.$$

(2) *For every* $f \in L^p(S, X)$, *there exists a closed separable subspace* Y *of* G *such that*

$$\pi(f(s)) \cap Y \neq \emptyset \qquad \textit{for a.e.} \quad s \in S.$$

Proof. Let $f \in L^p(S, X)$. Then f is strongly measurable. By assumption, there exists a closed and separable subspace Y of G and a null set N_1 in $S(\mu(N_1) = 0)$ such that

$$\pi(f(s)) \cap Y \neq \emptyset \qquad \text{for} \quad s \in S \backslash N_1.$$

By [7, Lemma 10.5], there exists a null set N_2 in S such that

$$f(S \backslash N_2) \text{ is a separable subset of } X.$$

Also, by [6, Lemma 2], there exists a null set N_3 in S such that

$$f\colon S \backslash N_3 \to X \text{ is weakly measurable.}$$

Since f represents an equivalence class of functions (the equivalence relation being "equality a.e.") we may assume, without loss of generality, that $f(s) = 0$ for all $s \in N_1 \cup N_2 \cup N_3$. Then $f\colon S \to X$ is strongly and weakly measurable, has separable range in X and, since $0 \in Y$,

$$\pi(f(s)) \cap Y \neq \emptyset \qquad \text{for all} \quad s \in S. \tag{4.1}$$

Assume for now that the mapping Φ_f defined by (2.5) admits a strongly measurable selection $g\colon S \to G$. Then

$$g(s) \in \pi(f(s)) \qquad \text{for a.e.} \quad s \in S, \tag{4.2}$$

and consequently, by [8, Theorem 6.1(b)],

$$\|g(s)\| \leqslant 2\, \|f(s)\| \qquad \text{for a.e.} \quad s \in S.$$

Hence, since $f \in L^p(S, X)$, we obtain that $g \in L^p(S, G)$ and, by (4.2), that

$$\begin{aligned} \|f - h\|_p &= \left\{ \int_S \|f(s) - h(s)\|^p \, d\mu \right\}^{1/p} \ (= \text{ess.sup}\, \|f(s) - h(s)\| \text{ if } p = \infty) \\ &\geqslant \left\{ \int_S \|f(s) - g(s)\|^p \, d\mu \right\}^{1/p} \ (\geqslant \text{ess.sup}\, \|f(s) - g(s)\| \text{ if } p = \infty) \\ &= \|f - g\|_p, \end{aligned}$$

for every $h \in L^p(S, G)$.

Therefore g is a best approximation of f in $L^p(S, G)$ and consequently, since f is arbitrary in $L^p(S, X)$, we obtain that $L^p(S, G)$ is proximinal in $L^p(S, X)$.

To finish the proof of the theorem, we need to show that the mapping Φ_f admits a strongly measurable selection.

Consider the set-valued mapping $\varphi: S \to 2^Y$ defined by

$$\varphi(s) = \pi(f(s)) \cap Y = \Phi_f(s) \cap Y. \tag{4.3}$$

Then, by (4.1), $\varphi(s)$ is a nonempty closed subset of Y for every $s \in S$. Hence, since Y is a separable Banach space, if we prove that φ is weakly measurable then, by the Kuratowski-Ryll-Nardzewski measurable selection theorem [7, p. 133], φ and consequently, by (4.3), Φ_f would admit a weakly measurable selection $g: S \to G$ with range contained in Y. This, with [7, Lemma 10.3] and the fact that Y is separable, would imply that g is also a strongly measurable selection for Φ_f which would complete the proof of the theorem.

Therefore, to complete the proof of the theorem, we now prove that φ is weakly measurable:

Let θ be an open set in Y and let R be as given in assumption (1) of the theorem. Since Y is separable, there exists a sequence $\{g_n\} \subset \theta$ and a sequence $\{r_n\} \subset (0, \infty)$ such that

$$\theta = \bigcup_n B(g_n, r_n) = \bigcup_n B(g_n, Rr_n). \tag{4.4}$$

We have, since $\theta \subset Y$,

$$\begin{aligned}\varphi^{-1}(\theta) &= \{s \in S: \varphi(s) \cap \theta \neq \varnothing\} \\ &= \{s \in S: \pi(f(s)) \cap \theta \neq \varnothing\} \\ &= \{s \in S: f(s) \in \theta + \hat{G}\} \\ &= f^{-1}(\theta + \hat{G}).\end{aligned}$$

But, by (4.4) and Lemma 4.2, we have

$$\begin{aligned}\theta + \hat{G} &= \bigcup_n B(g_n, r_n) + \hat{G} \subset \bigcup_n [\overline{B(g_n, r_n) + \hat{G}}] \\ &\subset \bigcup_n B(g_n, Rr_n) + \hat{G} = \theta + \hat{G}.\end{aligned}$$

Therefore, we obtain

$$\varphi^{-1}(\theta)=f^{-1}\left\{\bigcup_n [\overline{B(g_n, r_n)+\hat{G}}]\right\}$$
$$=\bigcup_n \{f^{-1}[\overline{B(g_n, r_n)+\hat{G}}]\}.$$

Consequently, since $f\colon S\to X$ is weakly measurable and $(\overline{B(g_n, r_n)+\hat{G}})$ is closed for all n, we obtain that $\varphi^{-1}(\theta)$ is measurable. The proof of the theorem is now complete. ∎

Remark 4.1. We note that if G is separable, then condition (2) of Theorem 4.1 is automatically satisfied. We also note that if ψ is lower semicontinuous then, by Corollary 3.1, condition (1) of Theorem 4.1 is satisfied. Therefore $L^p(S, G)$ is proximinal in $L^p(S, X)$, if G is separable and ψ is lower semicontinuous.

Finally we note that condition (2) of Theorem 4.1 is a necessary condition for $L^p(S, G)$ to be proximinal in $L^p(S, X)$, $1\leqslant p<\infty$. Indeed: If $f\in L^p(S, X)$ has a best approximation h in $L^p(S, G)$ then h is strongly measurable and, by [7, p. 115], there exists a null set N in S such that $h(S\backslash N)$ is separable. Let $\{g_n\}\subset h(S\backslash N)$ be dense in $h(S\backslash N)$ and let $Y=\overline{\operatorname{span}\{g_n\}}$. Then Y is a closed separable subspace of G and

$$h(S\backslash N)\subset Y.$$

This, with [6, Corollary 2], implies that

$$\pi(f(s))\cap Y\neq\varnothing \qquad \text{for a.e.} \quad s\in S.$$

Several old and new results can be obtained as corollaries of Theorem 4.1. We only include some of them in our next theorem:

THEOREM 4.2. *Let G be a proximinal subspace of X. Then, for $0<p\leqslant\infty$, $L^p(S, G)$ is proximinal in $L^p(S, X)$, if one of the following assumptions holds:*

(1) *G is convexly approximatively compact in X.*
(2) *G is reflexive.*
(3) *G has the C-property in X.*
(4) $\overline{\operatorname{span}\hat{G}}$ *is reflexive.*
(5) *G is separable and locally proximinal in X.*
(6) *X is a dual space and, G is $\|\cdot\|$-separable and Wk^*-closed in X.*
(7) *G is $\|\cdot\|$-separable and the pair (G, X) has property (HV).*

Proof. We only need to check that, if any of the above assumptions holds, then conditions (1) and (2) of Theorem 4.1 hold.

For assumptions (1)–(4), it follows from Theorem 3.1 that

$$B+\hat{G} \text{ is closed} \tag{4.5}$$

and that, for every closed separable subspace Y of G,

$$Y+\hat{G} \text{ is closed.} \tag{4.6}$$

From (4.5) we obtain that condition (1) of Theorem 4.1 holds and, from (4.6) and Lemma 4.1, we obtain that condition (2) of Theorem 4.1 holds. This ends the proof for assumptions (1)–(4).

For assumptions (5)–(7), if follows from Theorem 3.1 that equation (4.5) holds and consequently condition (1) of Theorem 4.1 holds. Since in assumptions (5)–(7) G is $\|\cdot\|$-separable, we obtain directly that condition (2) of Theorem 4.1 holds.

This completes the proof of the theorem. ∎

Finally, we close this section by giving a single condition that implies the proximinality of $L^p(S, G)$ in $L^p(S, X)$, given that G is proximinal in X. We have:

THEOREM 4.3. *Let G be a proximinal subspace of X. Then, for $0<p\leqslant\infty$, $L^p(S, G)$ is proximinal in $L^p(S, X)$, if $(Y\cap B)+\hat{G}$ is closed for every closed separable subspace Y of G.*

Proof. From the assumptions and Lemma 3.2 we obtain that equations (4.5) and (4.6) hold. From equation (4.5) we obtain that condition (1) of Theorem 4.1 holds and, from equation (4.6) and Lemma 4.1, we obtain that condition (2) of Theorem 4.1 holds. This completes the proof. ∎

5. FURTHER RESULTS

In this section S will denote the unit real interval and μ is the Lebesgue measure. For $1\leqslant p<\infty$, we introduce the following Banach spaces which will be called "Factor L^p-spaces":

$$L_0^p(S, X)=X,$$

$$L_n^p(S, X)=L^p(S, L_{n-1}^p(S, X)), \qquad n=1, 2, \ldots.$$

For $n\geqslant 1$ we introduce the following notations:

The norm in $L_n^p(S, X)$ is denoted by $\|\cdot\|_{p,n}$. If $L_n^p(S, G)$ is proximinal in $L_n^p(S, X)$, then we denote the metric projection by π_n and we let $\psi_n: L_n^p(S, X) \to [0, \infty)$ be defined by

$$\psi_n(f) = d_n(0, \pi_n(f)),$$

where d_n denotes the distance in $L_n^p(S, X)$.

Also, for $n \geqslant 1$, we set

$$B_n = \{g \in L_n^p(S, G): \|g\|_{p,n} \leqslant 1\}$$

and

$$\hat{G}_n = \{f \in L_n^p(S, X): d_n(f, L_n^p(S, G)) = \|f\|_{p,n}\}.$$

For $n = 0$ we keep the same notations introduced in Sections 1 and 2. Before we proceed, we note that $L_1^p(S, X) = L^p(S, X)$ and that $\|\cdot\|_{p,1} = \|\cdot\|_p$.

We now state our theorem concerning the proximinality of $L_n^p(S, G)$ in $L_n^p(S, X)$:

THEOREM 5.1. *Let G be a separable Chebyshev subspace of X and let $1 \leqslant p < \infty$. Suppose that ψ is lower semicontinuous. Then, for every $n \geqslant 1$, we have*

(i) *$L_n^p(S, G)$ is separable and Chebyshev in $L_n^p(S, X)$.*

(ii) *ψ_n is lower semicontinuous.*

(iii) *$B_n + \hat{G}_n$ is closed.*

Proof. We prove this by induction:

Case I: $n = 1$. (i) The proximinality of $L_n^p(S, G)$ in $L_n^p(S, X)$ follows from Remark 4.1. Since S is the unit real interval and G is separable, the separability of $L_1^p(S, G)$ follows by routine computations. The Chebyshevity of $L_1^p(S, G)$ in $L_1^p(S, X)$ now follows from [6, Corollary 2] and the Chebyshevity of G in X.

(ii) Let $f \in L_1^p(S, X)$ and let $\{f_n\} \subset L_1^p(S, X)$ be such that $\|f_n - f\|_{p,1} \xrightarrow[n]{} 0$. Then there exists a subsequence $\{f_{n_k}\}$ such that

$$\|f_{n_k}(s) - f(s)\|_{p,1} \xrightarrow[n]{} 0 \qquad \text{for a.e.} \quad s \in S.$$

Therefore, since ψ is lower semicontinuous and G is Chebyshev, we have

$$\|\pi(f(s))\| \leqslant \liminf_k \|\pi(f_{n_k}(s))\| \qquad \text{for a.e.} \quad s \in S. \tag{5.1}$$

By [6, Corollary 2] and the fact that G and $L_1^p(S, G)$ are Chebyshev in X and $L_1^p(S, X)$ respectively we have, for every $h \in L_1^p(S, X)$,

$$[\pi_1(h)](s) = \pi(h(s)) \qquad \text{for a.e.} \quad s \in S.$$

Hence, since $\pi_1(h) \in L_1^p(S, G)$, $\pi(h(s))$ is a strongly measurable function of s and we have

$$\|\pi_1(h)\|_{p,1} = \int_S \|\pi(h(s))\| \, d\mu, \qquad h \in L_1^p(S, X).$$

This, together with (5.1), implies that

$$\begin{aligned}\|\pi_1(f)\|_{p,1} &= \int_S \|\pi(f(s))\| \, d\mu \\ &\leqslant \int_S \liminf_k \|\pi(f_{n_k}(s))\| \, d\mu \\ &\leqslant \liminf_k \int_S \|\pi(f_{n_k}(s))\| \, d\mu \\ &= \liminf_k \|\pi_1(f_{n_k})\|_{p,1}.\end{aligned}$$

Noting that the above holds true if the sequence $\{f_n\}$ is replaced by any subsequence to start with, we obtain

$$\|\pi_1(f)\|_{p,1} \leqslant \liminf_n \|\pi_1(f_n)\|_{p,1}$$

or, in other words,

$$\psi_1(f) \leqslant \liminf_n \psi_1(f_n).$$

Therefore ψ_1 is lower semicontinuous.

(iii) Since ψ_1 is lower semicontinuous we obtain, by [5, p. 40], that $\psi_1^{-1}[0, 1]$ is closed.

But, since $L_1^p(S, G)$ in Chebyshev in $L_1^p(S, X)$, we have

$$\begin{aligned}\psi_1^{-1}[0, 1] &= \{f \in L_1^p(S, X) \colon \|\pi_1(f)\| \leqslant 1\} \\ &= \{f \in L_1^p(S, X) \colon \pi_1(f) \in B_1\} \\ &= B_1 + \hat{G}_1.\end{aligned}$$

Therefore, $B_1 + \hat{G}_1$ is closed.

Case II: $n \geqslant 2$. By case I, all the assumptions on G that were sufficient to establish case I are carried through to $L_1^p(S, G)$. Hence case II follows directly by induction. This completes the proof of the theorem. ∎

Acknowledgment

The author thanks Professor R. Khalil for his valuable comments during the course of this research.

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